Quantitative Evaluation Methods

- Random Variable X
  - a function that assigns a real number $X(s)$ to each sample point s in sample space S
  - e.g. coin toss, number of heads in a sequence of 3 tosses
    - »

s	hhh	hht	hth	htt	thh	tht	tth	ttt
$X(s)$	3	2	2	1	2	1	1	0

  - X is a random variable taking on values in the set

$$S_X = \{0,1,2,3\}$$

Quantitative Evaluation Methods

- Cumulative Distribution Function (cdf)
  - The cdf of a random variable X is defined as the probability of the event $\{X \leq x\}$

$$F_X(x) = P(X \leq x) \text{ for } -\infty < x < +\infty$$

$$F_X(x) = \text{prob. of event } \{s : X(s) \leq x\}$$

$$F_X(x) = \text{is a probability, i.e. } 0 \leq F_X(x) \leq 1$$

$$F_X(x) \text{ is monotonically non-decreasing,}$$
$$\text{i.e. if } x_1 \leq x_2 \text{ then } F_X(x_1) \leq F_X(x_2)$$

$$\lim_{x \to \infty} F_X(x) = 1 \qquad \lim_{x \to -\infty} F_X(x) = 0$$

Quantitative Evaluation Methods

- Probability Density Function (pdf)
  - The pdf of a random variable is the derivation of $F_X(x)$

$$f_X(x) = \frac{dF_X(x)}{dx}$$

  - Since $F_X(x)$ is a non-decreasing function,

$$f_X(x) \geq 0$$

Quantitative Evaluation Methods

- Expectation of a random variable
  - in order to completely describe the behavior of a random variable, an entire function, namely the cdf or pdf, must be given
  - however, sometime we are just interested in parameters that summarize information

$$E(X) = \int_{-\infty}^{\infty} x f_X(x) dx$$

i.e. mean time to failure = expected lifetime of the system

Reliability R(t)

- R(t) = probability that system is working at time t, and any time before that => [0,t]
- X = random variable representing life of system
- Let

$$N = \text{initial number of resources of a system}$$

$$N_o(t) = \text{number of resources operating at time t}$$

$$N_f(t) = \text{number of resources failed at time t}$$

Reliability R(t)

$$\begin{aligned} R(t) &= P(X > t) \\ &= 1 - P(X \le t) \\ &= 1 - Q(t) \\ &= 1 - \frac{N_f(t)}{N} \end{aligned}$$

$$\frac{dR(t)}{dt} = -\frac{1}{N}\frac{dN_f(t)}{dt}$$

$$\frac{dN_f(t)}{dt} = -N\frac{dR(t)}{dt}$$

instantaneous rate at which components are failing

$$\frac{dN_f(t)}{dt} = -N\frac{dR(t)}{dt} \qquad (1)$$

div by $N_o(t)$
to get

$$z(t) = \frac{1}{N_o(t)}\frac{dN_f(t)}{dt} \qquad (2)$$

this is called

hazard function
hazard rate
failure rate function

which is the normalized failure rate

using (1) in (2), i.e.

$$\frac{dN_f(t)}{dt} = -N\frac{dR(t)}{dt} \qquad z(t) = \frac{1}{N_o(t)}\frac{dN_f(t)}{dt}$$

we get

$$z(t) = -\frac{N}{N_o(t)}\frac{dR(t)}{dt}$$

expressed in terms of Reliability only with $R(t) = \frac{N_o(t)}{N}$

$$z(t) = -\frac{1}{R(t)}\frac{dR(t)}{dt}$$

expressed in term of unreliability Q(t)

$$
\begin{aligned}
z(t) &= -\frac{1}{R(t)}\frac{dR(t)}{dt} \\
&= -\frac{1}{1-Q(t)}\frac{d(1-Q(t))}{dt} \\
&= \frac{1}{1-Q(t)}\frac{dQ(t)}{dt}
\end{aligned}
$$

Result often used:

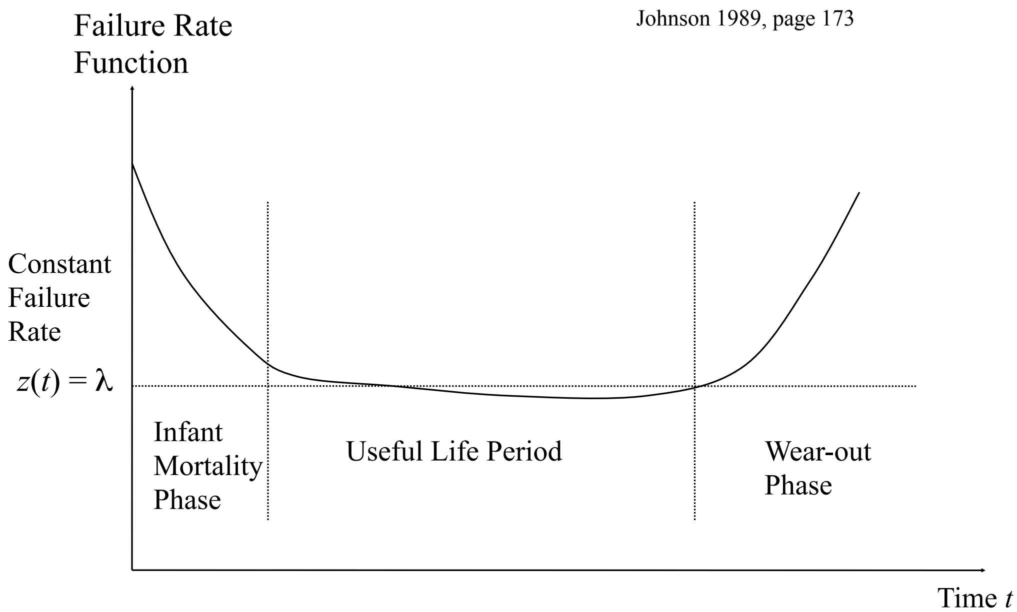
$$\frac{dR(t)}{dt} = -z(t)R(t)$$

Bathtub Curve

- Infant mortality phase
  - burn-in to bypass infant mortality
- Useful life period
- Wear-out phase
  - exchange before wear-out phase
- Therefore one may assume constant failure rate function *z*(t), i.e $z(t) = \lambda$

Bathtub Curve

Johnson 1989, page 173

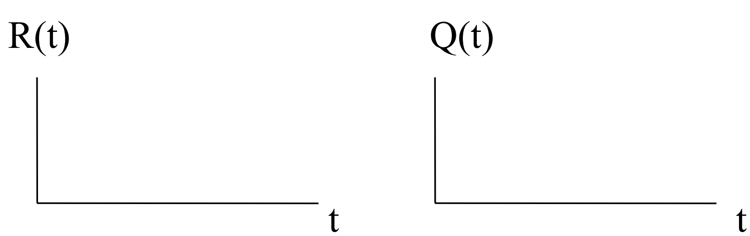


assuming constant z(t)

$$\frac{dR(t)}{dt} = -z(t)R(t)$$
$$= -\lambda R(t)$$

solving the differential equation we get

$$R(t) = e^{-\lambda t}$$



solving $$\frac{dR(t)}{dt} = -\lambda R(t)$$

$$\frac{R'(t)}{R(t)} = -\lambda$$

$$\int \lambda \, dt = -\int_0^t \frac{R'(t)}{R(t)} \, dt$$

$$= -\int_{R(0)}^{R(t)} \frac{dR}{R}$$

$$-\ln R(t) = \int_0^t \lambda \, dt$$

$$R(t) = e^{-\lambda t}$$

Mean Time to Failure (MTTF)

Expected lifetime

$$E[X] = \int_{-\infty}^{\infty} x f(x) dx$$

Mean Time to Failure

$$MTTF = \int_{-\infty}^{\infty} t f(t) dt$$

where $f(t)$ is the failure density function

$$f(t) = \frac{dQ(t)}{dt} = \frac{d(1-R(t))}{dt}$$

Mean Time to Failure (MTTF)

Now, we can rewrite

$$\frac{d(1-R(t))}{dt} = -\frac{dR(t)}{dt}$$

and use integration by parts

(recall) $\int u dv = uv - \int v du$

u and v are both functions of t

to get

$$MTTF = \int_0^\infty t\frac{Q(t)}{dt}dt = -\int_0^\infty t\frac{R(t)}{dt}dt = \left[-tR(t) + \int R(t)dt\right]_0^\infty = \int_0^\infty R(t)dt$$

Mean Time to Failure (MTTF)

Thus the expected lifetime is

$$E(t) = \int_0^\infty R(t)dt$$

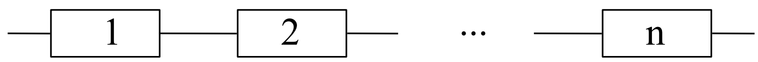
$$= \int_0^\infty e^{-\lambda t}dt$$

$$= \frac{1}{-\lambda} e^{-\lambda t}\Big|_0^\infty$$

$$= \frac{1}{\lambda}$$

Reliability of Series System

- Any one component failure causes system failure
- Reliability Block Diagram (RBD)



$$R(t)_{\text{series}} = \prod_{i=1}^{n} R_i(t)$$

$$= \prod_{i=1}^{n} e^{-\lambda_i t}$$

$$= e^{-(\sum_{i=1}^{n} \lambda_i)t}$$

Reliability of Series System

thus $$\lambda_{\text{series}} = \sum_{i=1}^{n} \lambda_{\text{i}}$$

Mean time to failure of series system:

$$MTTF_{\text{series}} = \frac{1}{\sum_{i=1}^{n} \lambda_{\text{i}}}$$

Thus the MTTF of the series system is much smaller than the MTTF of its components

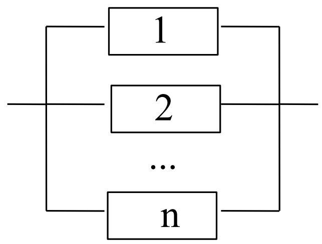
if $X_i \equiv$ lifetime of component i then

$0 \le E[X] \le \min\{E[X_i]\}$

system is weaker than weakest component

Reliability of Parallel System

- All components must fail to cause system failure
- Reliability Block Diagram (RBD)



  - assume mutual independence

X is lifetime of the system

$$X = \max\{X_1, X_2, \ldots, X_n\} \qquad \text{n components}$$

$$R(t)_{\text{parallel}} = 1 - \prod_{i=1}^{n} Q_i(t)$$

$$= 1 - \prod_{i=1}^{n} (1 - R_i(t))$$

$$\geq 1 - (1 - R_i(t))$$

Assuming all components have exponential distribution with parameter λ

$$R(t) = 1 - (1 - e^{-\lambda t})^n$$

$$E(X) = \int_0^{\infty} [1-(1-e^{-\lambda t})^n]dt$$

$$= \quad \ldots$$

$$= \frac{1}{\lambda} \sum_{i=1}^{n} \frac{1}{i}$$

Trivedi 1982, Page 218

$$\approx \frac{\ln(n)}{\lambda}$$

from previous page

$$Q(t)_{\text{parallel}} = \prod_{i=1}^{n} Q_i(t)$$

Product law of unreliability